A remark on Leclerc's Frobenius categories

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Abstract. Leclerc recently studied certain Frobenius categories in connection with cluster algebra structures on coordinate rings of intersections of opposite Schubert cells. We show that these categories admit a description as Gorenstein projective modules over an Iwanaga-Gorenstein ring of virtual dimension at most two. This is based on a Morita type result for Frobenius categories.

1 Motivation

Let G be a complex simple Lie group of type $Q = A, D$ or E (eg $G = \mathrm{SL}_{n+1}(\mathbb{C})$ for $Q = A_n$) with Borel subgroup $B \subset G$ (eg $B = \{\text{upper triangular matrices}\}$) and Weyl group W (eg $W \cong S_{n+1}$ given by permutation matrices).

For a Weyl group element $w \in W$ there are associated subvarieties C_w (*Schubert cell*) and C^w (*opposite Schubert cell*) in the flag variety G/B. On the other hand, there is a torsion pair $(\mathcal{C}_w, \mathcal{C}^w)$ in the category of finite dimensional modules over the preprojective algebra $\Pi := \Pi(Q)$ and the categories $\mathcal{C}_w$, $\mathcal{C}^w$ are Frobenius and have projective generators (in fact, the latter statements may be deduced from Proposition 5). These Frobenius categories were used by Geiß, Leclerc & Schröer to categorify cluster algebra structures on coordinate rings of the corresponding (opposite) Schubert cells [**3**].

Let $v \in W$. The intersections $C_{v,w} := C^v \cap C_w$ are known as *open Richardson varieties* and have been studied by Kazhdan-Lusztig in connection with KL-polynomials. Generalizing the aforementioned work [**3**], Leclerc [**5**] categorifies a cluster subalgebra of the coordinate rings of $C_{v,w}$ using the intersection $\mathcal{C}_{v,w}$ of a torsion free part $\mathcal{C}^v$ with a torsion part $\mathcal{C}_w$ of two torsion pairs mentioned above. Under some finiteness assumptions he obtains a cluster algebra structure on the whole coordinate ring and he conjectures that this holds in general.

The subcategories $\mathcal{C}_{v,w} \subseteq \mathsf{mod}\,\Pi$ inherit an exact structure which is again Frobenius.

Aim Explain this in a more abstract setting and give equivalent descriptions of $\mathcal{C}_{v,w}$.

This is summarized in the following Proposition which is a special case of Proposition 5.

Proposition 1 *Let* $\mathcal{C}_{v,w} := \mathcal{C}_w \cap \mathcal{C}^v \subseteq \mathsf{mod}\ \Pi$. *Then*

(a) $\mathcal{C}_{v,w}$ *is a Frobenius category with* $\mathsf{proj}\,\mathcal{C}_{v,w} = \mathsf{add}\, f_v t_w(\Pi) = \mathsf{add}\, t_w f_v(\Pi) =: \mathsf{add}\, P_{v,w}$. *Where* $t_u(-)$ *denotes the torsion radical and* $f_u(-) := (-)/t_u(-)$ *for a torsion pair* $(\mathcal{C}_u, \mathcal{C}^u)$.

(b) $\mathcal{C}_{v,w} \xrightarrow{\mathsf{Hom}_{\mathcal{C}_{v,w}}(P_{v,w},-)} \mathrm{GP}\big(\Pi_{v,w}\big)$ *is an exact equivalence, where* $\Pi_{v,w} := \mathsf{End}_{\mathcal{C}_{v,w}}(P_{v,w})$ *is an Iwanaga-Gorenstein ring of virtual dimension at most two.*

(c) *In particular,* $\mathcal{C}_{v,w}$ *is equivalent to the subcategory of second syzygies of finite dimensional* $\Pi_{v,w}$*-modules.*

(d) *The functors* f_v *and* t_w *induce ring homomorphisms* $\Pi_w := \mathsf{End}_{\mathcal{C}_w}(t_w(\Pi)) \to \Pi_{v,w}$ *and* $\Pi^v := \mathsf{End}_{\mathcal{C}^v}(f_v(\Pi)) \to \Pi_{v,w}$. *These are surjective if* $\mathcal{C}_v \subseteq \mathcal{C}_w$. *In turn, this condition is equivalent to* $w = v'v$ *with* $l(w) = l(v') + l(v)$, *called condition (P) in Leclerc* [**5**, 5.1].

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(e) *(see [**1**, 5.16]) If condition (P) holds, then $\Pi_{v,w}$ is Morita equivalent to $\Pi_{v'}$. Therefore, $\Pi_{v,w}$ has the same virtual dimension as $\Pi_{v'}$ which is at most* 1*,* [**2**].

Remark 2 Let $\Lambda_w := \Pi/I_w$ be the algebra considered in [**2**]. Then there are algebra isomorphisms $\Lambda_w \cong \Pi^{w_0 w^{-1}} \cong \Pi^{\mathrm{op}}_{w^{-1}}$, where w_0 denotes the longest Weyl group element.

2 A Morita type result for Frobenius categories

Definition/Proposition 3 A two-sided Noetherian ring R is called *Iwanaga-Gorenstein*, if $\mathsf{inj.dim}_R R < \infty$ and $\mathsf{inj.dim}\, R_R < \infty$. It is well-known that this implies $\mathsf{inj.dim}_R R = d = \mathsf{inj.dim}\, R_R$. We call $d =: \mathsf{vir.dim}\, R$ the *virtual dimension* of R.

In this case the category of *Gorenstein-projective* R-modules

$$\mathrm{GP}(R) := \{M \in \mathsf{mod}\, R \mid \mathsf{Ext}^i_R(M, R) = 0 \text{ for all } i > 0\}$$

is a Frobenius category with subcategory of projective-injective objects $\mathsf{proj}\, R$. Equivalently, $\mathrm{GP}(R)$ is the subcategory of d-th syzygies of finitely generated R-modules

$$\mathrm{GP}(R) \cong \Omega^d(\mathsf{mod}\, R) := \{\Omega^d(M) \mid M \in \mathsf{mod}\, R\}.$$

If R is a local commutative Noetherian ring, Gorenstein projective R-modules are precisely maximal Cohen-Macaulay R-modules and $\mathsf{inj.dim}_R R = \mathsf{kr.dim}\ R$.

Aim Characterize the categories of Gorenstein projective modules $\mathrm{GP}(R)$ over Iwanaga-Gorenstein rings R among all Frobenius categories.

Notation For an additive category $\mathcal{B}$, we denote by $\mathsf{mod}\,\mathcal{B}$ the category of finitely presented contravariant additive functors $\mathcal{B} \to \mathrm{Ab}$.

We first list properties of the categories $\mathcal{E} := \mathrm{GP}(R)$ for R Iwanaga-Gorenstein.

(i) $\mathsf{proj}\,\mathcal{E} = \mathsf{add}\, P\, (= \mathsf{proj}\, R)$ for some $P \in \mathcal{E}$ and $\mathsf{End}_{\mathcal{E}}(P)\, (\cong \mathsf{End}_R(R))$ is two-sided noetherian.

(ii) $\mathcal{E}$ is idempotent complete (since $\mathcal{E} \subseteq \mathsf{mod}\, R$ closed under direct summands).

(iii) $\mathcal{E}$ is Frobenius (use exact duality $\mathsf{Hom}_R(-, R)\colon\ \mathrm{GP}(R) \to \mathrm{GP}(R^{\mathrm{op}})$).

(iv) $\mathcal{E}$ has weak kernels and cokernels (use Auslander-Buchweitz approximation).

(v) $\mathsf{gl.dim\,mod}\,\mathcal{E}$, $\mathsf{gl.dim\,mod}\,\mathcal{E}^{\mathrm{op}} \leq n\, (= \max\{2, \mathsf{inj.dim}\, R\})$.

The following result may be interpreted as an analogue of Morita theory for Frobenius categories. The implication (b) $\Rightarrow$ (a) is well-known. The converse is the special case $\mathsf{proj}\,\mathcal{E} = \mathsf{add}\,\mathcal{P}, \mathcal{M} = \mathcal{E}$ of [**4**, 2.8], which is due to Iyama and inspired by a stable version of Dong Yang and the author [**4**, 2.15].

Proposition 4 *Let $\mathcal{E}$ be an exact category and let $P \in \mathcal{E}$. TFAE*

(a) *$\mathcal{E}$ and P satisfy the conditions (i)-(v) above.*

(b) *Set $R = \mathsf{End}_{\mathcal{E}}(P)$. $\mathsf{Hom}_{\mathcal{E}}(P, -)\colon \mathcal{E} \to \mathrm{GP}(R)$ is an exact equivalence and R is Iwanaga-Gorenstein with $\mathsf{vir.dim}\, R \leq \mathsf{gl.dim\,mod}\,\mathcal{E}$.*

3 From pairs of torsion pairs to Frobenius categories

Notation Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. In particular, there is a short exact sequence $0 \to t(X) \to X \to f(X) \to 0$ for all X in $\mathcal{A}$. This gives rise to functors $t\colon \mathcal{A} \to \mathcal{T}$ and $f\colon \mathcal{A} \to \mathcal{F}$, which are right (respectively left) adjoint to the canonical inclusions.

Proposition 5 *Let $\mathcal{A}$ be an abelian category with torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ and set $\mathcal{C}_{12} := \mathcal{T}_1 \cap \mathcal{F}_2$. Then the following statements hold:*

(a) *$\mathcal{C}_{12}$ is extension closed and idempotent complete, since $\mathcal{T}_1$ and $\mathcal{F}_2$ are. In particular, $\mathcal{C}_{12}$ inherits a natural exact structure from $\mathcal{A}$.*

(b) *$\mathcal{C}_{12}$ has kernels and cokernels. In other words, $\mathcal{C}_{12}$ is a preabelian category. In particular, the categories of finitely presented additive functors $\mathsf{mod}\,\mathcal{C}_{12}$ and $\mathsf{mod}\,\mathcal{C}_{12}^{\mathrm{op}}$ are abelian and have global dimension at most 2.*

For example, the composition of the canonical inclusions

$$t_1(\ker f) \hookrightarrow \ker f \hookrightarrow X \xrightarrow{f} Y$$

is a kernel of f. Here $\ker f$ denotes the kernel of f in $\mathcal{A}$.

(c) *If $\mathcal{T}_1$ has enough projectives and $\mathcal{F}_2$ has enough injectives, then $\mathcal{C}_{12}$ has enough injectives ($= t_1(\mathrm{inj}\ \mathcal{F}_2)$) and projectives ($= f_2(\mathsf{proj}\,\mathcal{T}_1)$).*

(d) *If additionally $\mathsf{Ext}^1_{\mathcal{C}_{12}}(X, Y) = 0 \Leftrightarrow \mathsf{Ext}^1_{\mathcal{C}_{12}}(Y, X) = 0$, then $\mathcal{C}_{12}$ is Frobenius. For example, this is satisfied if $\underline{\mathcal{A}}$ or $\mathcal{D}^b(\mathcal{A})$ are 2-Calabi-Yau. This in turn is known to hold for $\mathcal{A} = \mathrm{fdmod}(\widehat{\Pi(Q)})$, where Q is a quiver without loops and $\widehat{\Pi(Q)}$ is the m-adic completion of its preprojective algebra, where m denotes the ideal generated by all arrows.*

(e) *Assume additionally that $\mathsf{proj}\ \mathcal{T}_1 = \mathsf{add}\,P$ and $\mathrm{inj}\ \mathcal{F}_2 = \mathsf{add}\,I$, then $\mathsf{proj}\,\mathcal{C}_{12} = \mathsf{add}\,f_2(P) = \mathsf{add}\,t_1(I)$. We assume that $\Pi_{12} := \mathsf{End}_{\mathcal{C}_{12}}(f_2(P))$ is two-sided noetherian. Then there is an exact equivalence*

$$\mathcal{C}_{12} \xrightarrow{\mathsf{Hom}_{\mathcal{C}_{12}}(f_2(P), -)} \mathrm{GP}\big(\Pi_{12}\big),$$

and Π_{12} is Iwanaga-Gorenstein of virtual dimension at most 2.

(f) *In the situation of (e) the functors f_2 and t_1 induce ring homomorphisms $\varphi_2\colon \mathsf{End}_{\mathcal{T}_1}(P) \to \Pi_{12}$ and $\tau_1\colon \mathsf{End}_{\mathcal{F}_2}(I) \to \Pi_{12}$ with kernels given by the ideals of morphisms factoring over $t_2(P)$ and $f_1(I)$, respectively. The ring homomorphisms are surjective if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. In Example 7, φ_2 is injective but not surjective.*

Remark 6 This is an analogue of Buan, Iyama, Reiten & Scott's [**2**] dual description of Geiß, Leclerc & Schröer's categories $\mathcal{C}_w$ [**3**] as categories of submodules of projective modules over the algebra Λ_w, see also [**3**, Theorem 2.8]. Since Λ_w is Iwanaga-Gorenstein of virtual dimension 1, Gorenstein projective modules are first syzygies, which in turn are just submodules of projective modules. See also [**4**, Section 6] for a further discussion.

4 Examples, remarks and questions

Example 7 We consider the situation of [**5**, 3.16], i.e. Q is of type A_3, $w = s_1 s_3 s_2 s_1 s_3$ and $v = s_2$. Then $\varphi_2\colon \Pi_w := \mathsf{End}_{\mathcal{C}_w}(t_w(\Pi)) \to \Pi_{v,w}$ is injective and its cokernel in the category of vectorspaces is isomorphic to $\mathbb{C}$. Moreover, $\Pi_{v,w}$ is the Auslander algebra of the preprojective algebra of type A_2 and therefore is of global (and virtual) dimension 2.

Remark 8 (Duality) Let Q be a Dynkin quiver and let $D := \mathsf{Hom}_k(-, k)$ be the standard duality. It is well-known that there is an algebra isomorphism $\psi\colon \Pi \cong \Pi^{\mathrm{op}}$, which gives rise to a duality $\Phi\colon \mathsf{mod}\,\Pi \xrightarrow{D} \mathsf{mod}\,\Pi^{\mathrm{op}} \xrightarrow{\psi_*} \mathsf{mod}\,\Pi$. Using the notation in Leclerc [**5**, §3.2], one can check that $\Phi(P_{v,w}) \cong P_{w_0^{-1} w, w_0 v}$ holds, where w_0 denotes the longest Weyl group element. In particular, Φ induces an algebra isomorphism $\Pi_{v,w} \cong \Pi^{\mathrm{op}}_{w_0^{-1}w,w_0v}$. Thus $\Pi^v \cong \Pi_{v,w_0} \cong \Pi^{\mathrm{op}}_{\mathrm{id},w_0v} \cong \Pi^{\mathrm{op}}_{w_0v}$ for the algebras appearing in Proposition 1 (d).

Open Problem 9 Give a 'combinatorial description' of $\Pi_{v,w}$, eg as quiver with relations.

Remark 10 The number of isoclasses of indecomposable projective $\Pi_{v,w}$-modules seems to be unknown in general. It is not always bounded above by $|Q_0|$, see Example 7.

Question 11 (Leclerc) How does the virtual dimension of $\Pi_{v,w}$ depend on Q, v, w and (how) is this number related to the geometry of the open Richardson variety $C_{v,w}$?

Partial Answer 12 By Remark 2 and [**2**], $\mathsf{vir.dim}\,\Pi^v, \Pi_w \leq 1$. They are zero iff $\mathcal{C}^v$ (respectively, $\mathcal{C}_w$) are exact abelian subcategories of $\mathsf{mod}\,\Pi$, which are then equivalent to $\mathsf{mod}\,\Pi/e$ ($e \in \Pi$ idempotent). Thus if $\mathsf{vir.dim}\,\Pi^v, \Pi_w = 0$, then $\mathsf{vir.dim}\,\Pi_{v,w} = 0$ (since $\mathcal{C}_{v,w}$ is abelian). If one of Π^v and Π_w has virtual dimension zero, then $\mathcal{C}_{v,w}$ is the torsion (or torsion-free) part of a torsion pair in $\mathsf{mod}\,\Pi/e$. By Mizuno [**6**] and [**2**], $\Pi_{v,w} \cong (\Pi/e)^{v'}$ or $\cong (\Pi/e)_{w'}$ and is therefore of virtual dimension ≤ 1. Also $\mathsf{gl.dim}\,\Pi_v = n \leq 1$ (or $\mathsf{gl.dim}\,\Pi^w = m \leq 1$) implies $\mathsf{gl.dim}\,\Pi_{v,w} \leq \min\{n, m\}$. If both Π_v, Π^w have infinite global dimension and virtual dimension 1, then virtual dimensions $0, 1, 2$ occur for $\Pi_{v,w}$.

Remark 13 (Commutativity) It follows from work of Mizuno [**6**], that all torsion pairs in $\mathsf{mod}\,\Pi$ are of the form $(\mathcal{C}_w, \mathcal{C}^w)$ for some Weyl group element w. In particular, there are only finitely many torsion pairs, which is very surprising given the size of $\mathsf{mod}\,\Pi$. The explicit description of the associated functors t_w and f_w (see eg Leclerc [**5**, §3.2]) shows that $f_v t_w(M) \cong t_w f_v(M)$ for Weyl group elements $v, w \in W$ and $M \in \mathsf{mod}\,\Pi$. This seems very unusual for a pair of torsion pairs in general abelian categories and fails already for $\mathsf{mod}\,U_2(k)$, where $U_2(k)$ denotes the ring of 2×2 upper triangular matrices.

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